ON THE STATE OF STRESS IN AN ELASTIC BODY WITH A PLANE CRACK (CUT)

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In this paper we shall apply the methods of solution of three-dimensional contact problems of the theory of elasticity to the solution of problems of a body with a plane crack (cut).

Assume that the solution of the problem of the indentation of a frictionless punch into an elastic half-space is known. This means that a function $U(x, y, z)$, which is harmonic in the half-space $z \geqslant 0$, goes to zero at infinity and satisfies the following mixed boundary conditions:

(1) inside S, the area of contact between the punch and the boundary $z = 0$

$$U(x, y, 0) = b + \alpha x + \beta y - \varphi(x, y) \tag{1}$$

(2) outside the area S of the boundary $z = 0$

$$\partial U / \partial z = 0 \tag{2}$$

is given where $z = \varphi(x, y)$ is the equation of the base area of the punch and $b + \alpha x + \beta y$ is the rigid-body displacement of the punch.

The normal stress under the punch is determined from the following formula [1]

$$\sigma_z = -[mG / (m-1)] (\partial U / \partial z)_{z=0} \tag{3}$$

Here m is the reciprocal of the Poisson coefficient.

It is assumed that the function $\varphi(x, y)$ has continuous partial derivatives up to and including second order and that the boundary line of the punch does not press onto the elastic half-space. Thus, it follows from (3) that the function

$$V = \partial U / \partial z \tag{4}$$

which is harmonic in the half-space $z \geqslant 0$ will be continuous at $z = 0$. Since $U(x, y, z)$ satisfies the Laplace equation $\nabla^2 U = 0$, using (4) we obtain

$$\partial V / \partial z = \partial^2 U / \partial z^2 = -(\partial^2 U / \partial x^2 + \partial^2 U / \partial y^2) \tag{5}$$

When we utilize (1), (2), (4) and (5) we find that the function (4), which is harmonic in the half-space $z = 0$ and goes to zero at infinity, satisfies the following conditions at the boundary of the half-space $z = 0$: $V(x, y, 0) = 0$ outside the area S, and $\partial V/\partial z = \nabla^2 \varphi(x, y)$ inside the area S, where ∇^2 is the Laplace operator.

If we substitute

$$p(x, y) = mG \nabla^2 \varphi(x, y) / (m - 1) \tag{6}$$

then the harmonic function $V(x, y, z)$ will represent the solution of a problem on the stress-strain state of an unbounded elastic body with a plane crack S, whose surface is subject to a normal (compressive) stress $\sigma_z(x, y, +0) = \sigma_z(x, y, -0) = -p(x, y)$. The equation of the surface of the crack, which is expanded by the action of the normal pressure $p(x, y)$, is of the form

$$z = \pm V(x, y, 0), \quad \text{or} \quad z = \mp (m - 1) \sigma_z(x, y) / mG \tag{7}$$

Equations (3) and (4) were being considered here.

If the harmonic function $U(x, y, z)$ is the potential of a straight layer with a density $\gamma(x, y)$ then

$$\sigma_z = -2\pi mG\gamma(x, y) / (m - 1)$$

Consequently, equation (7) for the expended crack becomes

$$z = \pm 2\pi\gamma(x, y)$$

The displacement vector and the stress tensor, which exist in an unbounded elastic body with a plane crack S whose surface is subjected to a normal pressure $p(x, y)$ can be expressed in terms of $V(x, y, z)$ by means of known formulas.

Examples

1. If $\quad p(x, y) = \dfrac{mG}{m-1} \sum\limits_{i+j=0}^{n} [(i+2)(i+1) b_{i+2} + (j+2)(j+1) b_{i, j+2}] x^i y^j$

and the crack S is elliptic $x^2/a^2 + y^2/a^2(1 - k^2) \leqslant 1$, then using (7) and the author's results from another publication [2] for a punch with a surface

$$z = \sum_{i+j=2}^{n+2} b_{ij} x^i y^j$$

we obtain the equation for the expended crack

$$z = \mp 2\pi \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{a^2(1-k^2)}} \sum_{i+j=0}^{n} c_{ij} x^i y^j$$

Here the coefficients c_{ij} are found from the solution of a determinant of a set of linear algebraic equations whose coefficients are expressed in terms of elliptic integrals of the first and second kind.

2. If the pressure p_0 is constant and the crack is elliptic, then the equation of the expanded crack takes on the form of a tri-axial ellipsoid

$$x^2/a^2 + y^2/a^2(1-k^2) + z^2/b^2 = 1 \qquad (b = \pi(m-1) p_0 d_{00}^{20} / mG (d_{00}^{20} + d_{00}^{02}))$$

The values of d_{00}^{20} and d_{00}^{02} are given in the author's monograph [2].

3. If

$$p(x, y) = mG(e_{00} + e_{10}x + e_{01}y + e_{20}x^2 + e_{11}xy + e_{02}y^2)/(m-1)$$

the crack is circular of radius a, then, using (7) and the result of [2], we obtain

$$z = \pm \frac{2}{9\pi} \sqrt{a^2 - \rho^2} \Big[e_{00} + a^2(e_{20} + e_{02}) + \tfrac{2}{3} e_{10}x + \tfrac{2}{3} e_{01}y + \tfrac{2}{45}(11e_{02} - e_{20})x^2 + + \tfrac{8}{15} e_{11}xy + \tfrac{2}{45}(11e_{20} - e_{02})y^2 \Big]$$

4. If the pressure p_0 is constant, then, letting $b = 2(m-1)p_0 a/\pi mG$, we obtain Sneddon's result [3] on the form $x^2/a^2 + y^2/a^2 + z^2/b^2 = 1$ of a circular crack which is expanded by means of internal pressure p_0.

Note that the methods of solution of the problems of an elastic body with a plane crack can be applied to the solution of contact problems of the theory of elasticity under the condition that the boundary line of the punch does not press upon the elastic half-space, i.e. the normal stress at the contour of S, the area of contact between the punch and the half-space, goes to zero. Actually, if the function $V(x, y, z)$ represents the solution of the problem of an unbounded elastic body with a plane crack S whose surface is subjected to a normal pressure (6), then, using (4), the normal stress (3) inside S, the area of contact of an elastic half-space with a punch that has a base area $z = \varphi(x, y)$ is expressed in the form

$$\sigma_z = -mGV(x, y, 0)/(m-1) \tag{8}$$

Example

If we use Sneddon's results

$$V(x, y, 0) = \frac{2a(m-1)}{\pi mG} \int_{\rho/a}^{1} \frac{y\,dy}{(y^2 - \rho^2/a^2)} \int_{0}^{1} \frac{up(ayu)}{(1-u^2)^{1/2}}\,du \tag{9}$$

when a normal pressure $p(\rho)$ is applied to a circular crack of radius a, then having used (6), (8) and (9) we obtain the following formula for the normal stress under an axisymmetric punch

$$\sigma_z = -\frac{2amG}{\pi(m-1)} \int_{\rho/a}^{1} \frac{y\,dy}{(y^2 - \rho^2/a^2)^{1/2}} \int_{0}^{1} \frac{u\psi(ayu)}{(1-u^2)^{1/2}}\,du, \qquad \psi(ayu) = \nabla^2\varphi(\rho)\,|_{\rho = ayu} \tag{10}$$

With $z = A\rho^2$ Formula (10) leads to the Hertz result: for $z = A\rho$ we obtain the result of A. Love for a conical punch.

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